# Comment on 'Hamiltonian formulation for the theory of gravity and canonical transformations in extended phase space' by T P Shestakova

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## Abstract

We argue that the conclusion, 'we cannot consider the Dirac approach as fundamental and undoubted', made in the paper by Shestakova (*Class. Quantum Grav.* **28** 055009, 2011), is based upon an incomplete and flawed analysis of the simple model presented in section 3 of the article. We re-examine the analysis of this model and find that it does not support the author's conclusion. For the theory of gravity neither the equivalence of the effective action nor its Hamiltonian formulation is given by the author, therefore, we only provide a brief commentary.

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We examine the analysis of the simple Lagrangian, which was used in section 3 of [1] to illustrate that '[Dirac's] algorithm fails to produce correct results' for an arbitrary parametrisation. Two parametrisations (equations (S12) and (S13))[1] are discussed:

$$L_1 = -\frac{1}{2}\frac{a\dot{a}^2}{N} + \frac{1}{2}Na \; ; \qquad L_2 = -\frac{1}{2}\frac{a\dot{a}^2}{\sqrt{\mu}} + \frac{1}{2}\sqrt{\mu}a, \qquad N = \sqrt{\mu}. \tag{1}$$

In [1], Dirac's algorithm[2] is applied to $L_1$ and $L_2$ to build the gauge generators and then to find the corresponding gauge symmetries. For $L_2$, the generator, (S26), gives

$$\delta_2\mu = -\frac{1}{2\mu}\dot{\mu}\theta_2 + \dot{\theta}_2 \; , \qquad \delta_2 a = \frac{1}{2}\frac{\dot{a}}{\mu}\theta_2 \; ; \tag{2}$$

where $\theta_2$ is a gauge parameter. In addition to $\delta_2\mu$ (see equation (S27)), which was deemed in [1] to be the correct transformation, we used equation (S26) to obtain $\delta_2 a$. By applying the same method to $L_1$, another generator is found, (S32), which leads to the following transformations:

$$\delta_1 N = \dot{\theta}_1 \; , \quad \delta_1 a = \frac{\dot{a}}{N}\theta_1 \; , \tag{3}$$

which were not reported in [1]. Instead, it was declared that Dirac's method does not produce a 'correct' result for the $L_1$ parametrisation, and must be abandoned in favour of another method, the Extended Phase Space (EPS) approach.

Transformations (2) and (3) are written for different variables; therefore, to compare them we shall use $N = \sqrt{\mu}$. The transformations of the fields in $L_2$ under $\delta_1$ are

$$\delta_1\mu = 2N\delta_1 N = 2\sqrt{\mu}\dot{\theta}_1 \; , \quad \delta_1 a = \frac{\dot{a}}{\sqrt{\mu}}\theta_1 \; ; \tag{4}$$

similarly, the transformations of the fields of $L_1$ under $\delta_2$ are

$$\delta_2 N = \delta_2\sqrt{\mu} = \frac{1}{2\sqrt{\mu}}\delta_2\mu = -\frac{1}{2N^2}\dot{N}\theta_2 + \frac{1}{2N}\dot{\theta}_2 = \left(\frac{1}{2N}\theta_2\right)_{,0} \; , \quad \delta_2 a = \frac{1}{2}\frac{\dot{a}}{N^2}\theta_2 \; . \tag{5}$$

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[1] Equations indicated as (S##) are from [1].

[2] When we refer to Dirac's algorithm, we mean that all steps are performed: from introducing momenta for *all* variables, to finding gauge invariance using, for example, Castellani's algorithm [2], based on the Dirac conjecture that all of the first class constraints are needed to derive gauge transformations [3].

Hence transformations (2) and (3) are different for both sets of fields (compare (2) with (4) and (3) with (5)).

If transformation (2) (the 'correct' one) is applied to $N$, we obtain (5); this variation differs from equation (S33), which is reported in [1] to be the expected result. Let us designate equation (S33) as the 'second correct' result, and associate it with a transformation $\delta_3$.

What is the meaning of these various transformations? For a Lagrangian with a gauge symmetry, according to Noether's second theorem [4], there is a corresponding combination of Euler-Lagrange derivatives (ELD) – a differential identity (DI). If a transformation is known, the corresponding DI can always be restored [5]. For example, a DI for $L_2$ can be found from

$$\int \left[ E^{(2)}_{(\mu)} \delta_2 \mu + E^{(2)}_{(a)} \delta_2 a \right] dt = \int I^{(2)} \theta_2 dt \ , \tag{6}$$

where $E^{(2)}_{(\mu)} = \frac{\delta L_2}{\delta \mu}$ and $E^{(2)}_{(a)} = \frac{\delta L_2}{\delta a}$ are ELDs of $L_2$. Substituting the known gauge transformations (2) and performing simple rearrangements, we obtain

$$I^{(2)} = -\frac{1}{2\mu} \dot{\mu} E^{(2)}_{(\mu)} - \dot{E}^{(2)}_{(\mu)} + \frac{1}{2} \frac{\dot{a}}{\mu} E^{(2)}_{(a)} \equiv 0. \tag{7}$$

Similarly, for $L_1$ the DI is

$$I^{(1)} = -\dot{E}^{(1)}_{(N)} + \frac{\dot{a}}{N} E^{(1)}_{(a)} \equiv 0. \tag{8}$$

These results can be directly verified by substituting the corresponding ELDs or by performing transformations of the Lagrangian (e.g. $\delta_1 L_1 = \partial_0 \left( -\frac{a\dot{a}^2}{2N^2} \theta_1 + \frac{1}{2} a \theta_1 \right)$); thus confirming that Dirac's algorithm correctly finds a symmetry of the Lagrangian. In the Lagrangian approach, if one DI is found (e.g. using Dirac's algorithm), we can build more DIs and find new gauge symmetries by repeating steps (6)-(7) in inverse order. For example, let us modify DI (7)

$$\tilde{I}^{(2)} = 2\sqrt{\mu} I^{(2)} = -\partial_0 \left( 2\sqrt{\mu} E^{(2)}_{(\mu)} \right) + \frac{\dot{a}}{\sqrt{\mu}} E^{(2)}_{(a)} \equiv 0; \tag{9}$$

the transformations that this DI produces are the same as (4), so this is also symmetry of $L_2$. Similarly, considering

$$\tilde{I}^{(1)} = \frac{1}{2N} I^{(1)} = -\frac{1}{2N} \dot{E}^{(1)}_{(N)} + \frac{\dot{a}}{2N^2} E^{(1)}_{(a)} \equiv 0, \tag{10}$$

we obtain transformations (5). So, symmetries (2) and (5) for the 'correct' expressions and those for the 'incorrect' expressions, (3) and (4), are symmetries for both Lagrangians. More symmetries can be found by further modification of the DIs; and many parametrisations of a Lagrangian can be explored. For any symmetry specified, we can find a parametrisation for which Dirac's algorithm will lead to this same symmetry; e.g. for the 'second correct' symmetry the parametrisation is:

$$N = e^{-\varkappa} , \qquad L_3 = -\frac{1}{2}e^{\varkappa}a\dot{a}^2 + \frac{1}{2}e^{-\varkappa}a .$$ (11)

Repeating Dirac's analysis, as was done in [1] for (1), one obtains:

$$\delta_3\varkappa = -\dot{\varkappa}\theta_3 + \dot{\theta}_3 , \qquad \delta_3 a = -\dot{a}\theta_3 \quad \text{and} \quad \delta_3 N = -\dot{N}\theta_3 - N\dot{\theta}_3 , \qquad \delta_3 a = -\dot{a}\theta_3 .$$ (12)

So, the parametrisation of $L_3$ (not $L_2$) leads to equation (S33) – the 'second correct' symmetry.

The justification to call transformations (2), from the application of Dirac's method to $L_2$, 'correct' is based on an 'interpretation' of the field $\mu$ as the component $g_{00}$ of the metric tensor and on its invariance under diffeomorphism (see (S28), (S29)), which is known from the Einstein-Hilbert (EH) (not $L_2$) action; the components of the vector gauge parameter $\theta^\lambda$ must be carefully crafted: $\theta^0 = \frac{\theta_2}{2\mu}$ and $\theta^k = 0$. This approximation must be applied to all fields of a given model if one expects this 'diffeomorphism' to be a symmetry of $L_2$. The transformation of a scalar under diffeomorphism is known; and the same approximation leads to

$$\delta_{diff} a = -a\partial_\mu\theta^\mu \Longrightarrow \delta a = -a\partial_0\left(\frac{\theta_2}{2\mu}\right) = \frac{a\dot{\mu}}{2\mu^2}\theta_2 - \frac{a}{2\mu}\dot{\theta}_2 .$$ (13)

For $a$, there are no time derivatives of the gauge parameters in (2), (3), or in (12); therefore, the rationale for choosing this 'correct' transformation is based on a questionable 'interpretation' and 'approximation', which is not internally consistent.

The three examples considered here illustrate the equivalence of the Lagrangian and Hamiltonian methods for systems with gauge invariance, and show that all Lagrangian symmetries can also be derived using the Hamiltonian approach. The failure to find a parametrisation (as $L_3$) to derive a particular symmetry in the Hamiltonian approach is not

a failure of Dirac's method, and it is not a strong enough justification to advocate the use of a new approach: EPS or any other. For these examples, all symmetries can be derived in both approaches (Lagrangian and Hamiltonian). The question of which symmetry is 'correct', is beyond the realm of Lagrangian and Hamiltonian equivalence; it cannot be answered by performing a 'canonization' of a symmetry one expects[3]; a mathematical criterion is required.

In the case of the EH action we analysed the group properties of various symmetries [6]. Let us do the same for this model by calculating the commutators of different transformations. For $\delta_1$, the transformations which were not included in [1] (perhaps because they were deemed incorrect), we obtain

$$[\delta_1'', \delta_1'] = (\delta_1'' \delta_1' - \delta_1' \delta_1'') ((N, a), (\mu, a), (\varkappa, a)) = 0. \tag{14}$$

This is the simplest possible result (as in the Maxwell theory). For $\delta_2$, the 'correct' transformations, the result is

$$[\delta_2'', \delta_2'] (\mu, a) = \delta_2''' (\mu, a), \qquad \theta_2''' = \frac{1}{2\mu} \left[ \theta_2'' \dot{\theta}_2' - \theta_2' \dot{\theta}_2'' \right], \tag{15}$$

which has a field-dependent 'soft algebra' structure[4]. In such a case it might be possible that the Jacobi identity is not satisfied (i.e. failure to form a group); to check this, the evaluation of double commutators is needed, as was performed for the ADM transformations in [6]. Group properties notwithstanding, this 'correct' symmetry (15) is more complicated than the 'incorrect' one (14). For $\delta_3$, the 'second correct' symmetry, the commutator is

$$[\delta_3'', \delta_3'] ((N, a), (\mu, a), (\varkappa, a)) = \delta_3''' ((N, a), (\mu, a), (\varkappa, a)), \qquad \theta_3''' = \dot{\theta}_3'' \theta_3' - \dot{\theta}_3' \theta_3'', \tag{16}$$

which is simpler than (15), but not as simple as the 'incorrect' (14).

The EPS approach, designed to fix the 'failure' of Dirac's method, was applied to this simple model in section 4 of [1] to illustrate its advantages. Long calculations were performed

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[3] There may be a specific exception. In covariant theories one should expect a covariant result and expect that a covariant parametrisation would be preferable for the Hamiltonian, or as in the example considered, due to the simplicity of its Lagrangian.

[4] For the other two pairs, the field dependence is different, but consistent with field redefinitions $(N, a)$ and $(\varkappa, a)$ : $\frac{1}{2N^2}$ and $\frac{1}{2} e^{2\varkappa}$ in (15), instead of $\frac{1}{2\mu}$.

and 'for the original variable $N$ one gets the transformation [(S33)]' – the 'second correct' transformation. This outcome demonstrates that the author's approach is not a reliable algorithm. It focuses on an *a priori* 'canonization' of one symmetry ($\delta_2$); yet the author's method is insensitive to this choice of symmetry; instead, it successfully produces $\delta_3$ without contradiction.

A second advantage of the proposed method, emphasized by the author of [1], is that it supports the canonical structure of Poisson Brackets (PBs) in the *extended phase space*. For the three parametrisations considered, $(N, \pi_N)$, $(\mu, \pi_\mu)$, $(\varkappa, \pi_\varkappa)$ (the second pair, $(a, p)$, is decoupled), all PBs are canonical *without EPS* because of the following relations:

$$N = \sqrt{\mu}, \quad \pi_N = 2\sqrt{\mu}\pi_\mu \text{ and } N = e^{-\varkappa}, \quad \pi_N = -e^\varkappa \pi_\varkappa .$$

The canonicity of the PBs is a necessary, but not sufficient condition for there to be an equivalence of constrained Hamiltonians (see [7, 8]). The EPS approach is based on the choice of a 'correct' symmetry. And if it is known, then there is no need to use Dirac's method to confirm it. But if a symmetry is unknown, then Dirac's method (because it is parametrisation-dependent) allows one to find different symmetries of a Lagrangian and, at the same time, find the simplest, 'canonical', symmetry (not necessarily the 'canonized' one) and the unique 'canonical parametrisation' for which the Hamiltonian gives this symmetry. For example, if one were to apply Dirac's approach to $L_2$ or $L_3$ to find the simplest parametrisation, one would uniquely obtain $L_1$. Of course, for this simple model, this outcome is obvious from the inspection of the fields in the Lagrangian; the presence of fields in combinations $\sqrt{\mu}$ or $e^{-\varkappa}$ naturally suggests calling them $N$. Such a redefinition gives the natural parametrisation for this model; and the Hamiltonian will lead to the simplest symmetry, the symmetry that was rejected in [1].

For more complicated theories the simplest reparametrisation is not obvious, and a search can be difficult. Consider the ADM Lagrangian without any *a priori* knowledge of gauge symmetry ('correct' or 'incorrect'). Application of Dirac's method leads to transformations that do not form a group, making it necessary to find other parametrisations. This procedure can be formulated as an algorithm (we did not guess $L_3$). We have not applied it to the ADM Lagrangian; but we conjecture it will lead to a unique symmetry and a unique parametrisation: diffeomorphism invariance, and the metric tensor in which the EH action was originally written. Of these variables, which is more 'preferable because of its

geometrical interpretation' [1]: Einstein's metric or the ADM variables?

In the author's article the application of the EPS approach to GR is incomplete (the equivalence of the effective and EH actions is not demonstrated, and the Hamiltonian not given); and the only result that was obtained is a proof of the canonical relations of PBs for the class of parametrisations (S9) and (S73). Again, this does not require EPS variables; and for the case of ADM variables, this result was given in [7] to illustrate that having canonical PBs is not sufficient for two Hamiltonians to be equivalent.

## I.   ACKNOWLEDGMENT

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